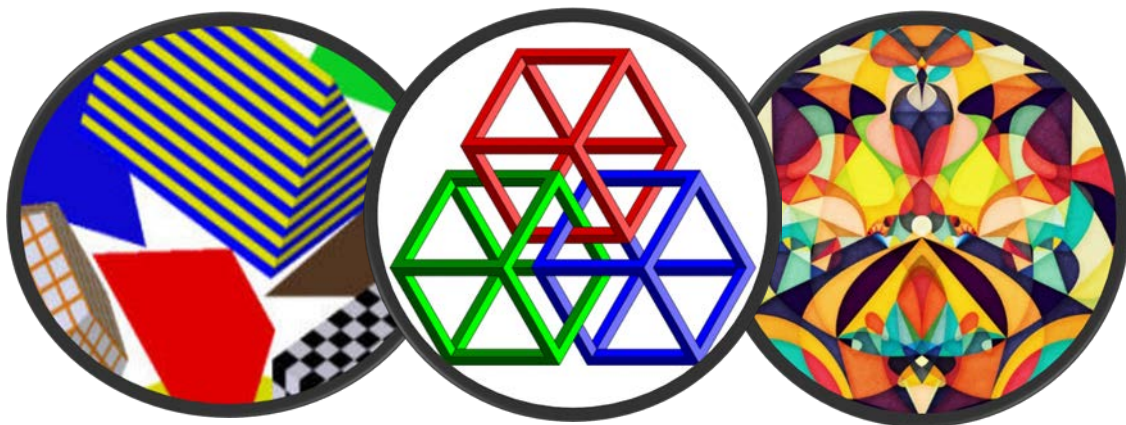


Maths Module 7

Geometry

This module covers concepts such as:

- two and three dimensional shapes
- measurement
- perimeter, area and volume
- circles



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Module 7

Geometry

1. Introduction
2. Two Dimensional Shapes
3. Area and Perimeter
4. Circles
5. Three Dimensional Shapes
6. Measurement
7. Area and Volume (extension)
8. Answers
9. Helpful Websites

1. Introduction

Geometry exists in the world around us both in manmade and natural form. This first section covers some aspects of **plane geometry** such as lines, points, angles and circles.

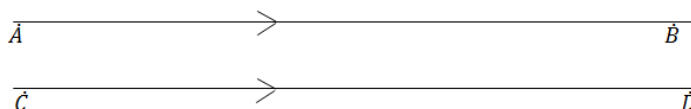
A **point** marks a point, a position, yet it does not have a size. However, physically, when we draw a point it will have width, but as an abstract concept, it really represents a point in our imagination.

A **line** also has no width; it will extend infinitely in both directions. When drawing a line, physically it has width, yet it is an abstract representation in our minds. Given two distinct points for a line to pass through – there will only ever be the one line. The points are generally referred to with upper case letters and the line with lower case. For instance we may draw line m through points A and B :



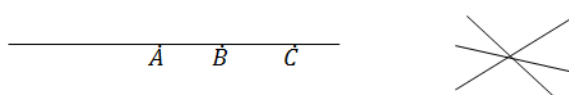
The line may be referred to as line AB or as line m . The distance between A and B is called a **line segment** or **interval**.

If we had two lines, there would be two possibilities. They would either meet at a single point, and they would form angles at the intersection. Or they would never meet, and then they would be **parallel** lines.



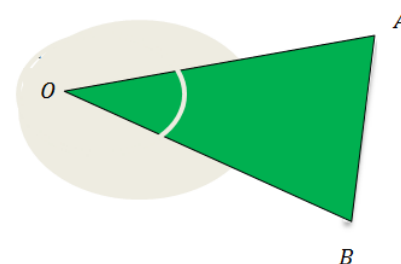
To indicate the lines AB and CD are parallel lines we write: $AB \parallel CD$

If there are three or more points on a straight line, they are **collinear** and three or more lines that meet at the same intersecting point are called **concurrent**.



If we look at the line connecting points ABC , we can describe a few things. For example, the point A divides the line (endpoint to endpoint) into two pieces and these are called **rays**. The ray AC contains the points A, B & C .

Angles are a measure of rotation between two **rays**: OA and OB . Hence the angle would be called angle AOB or BOA and the angle sign \angle so $\angle AOB$



This angle outside of the rays (the shaded area) is a **reflex** angle. However, unless specified as 'reflex,' $\angle AOB$ would indicate the angle between the rays.

Hence, the size of the angle is measured by the amount of turn between the two rays. Imagine if you stood at O and faced A , then you turned to face B , the amount of **rotation** is the size of the angle. If we turned the other way then we could measure the rotation of the reflex angle. Angles are measured in **degrees** ($^\circ$) and there are 360 degrees in one full turn. When two rays point in the opposite direction we have a straight angle which is 180° , half of a full turn.

Two angles that add up to 180° are called **supplementary** angles and two angles that add up to 90° are called **complementary** angles.

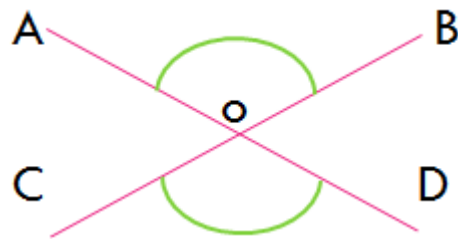
1. Your Turn:

a. Write down a pair of supplementary angles, and where they may exist on the diagram below.

b. Write down two pairs of complementary angles.

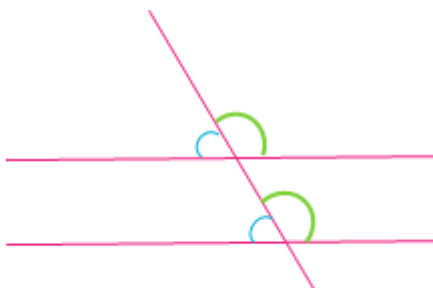
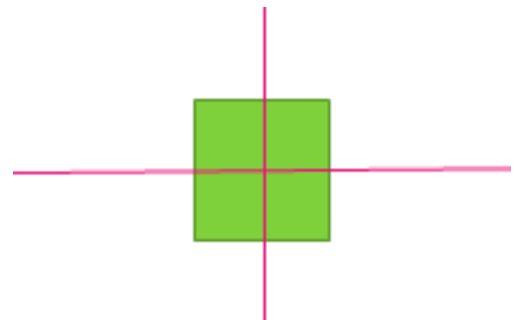
c. Write down four angles that add up to 360° and mark where they might exist on the diagram below.

d. What can you say about vertically opposite angles?



More about lines and angles

The lines on the image (right) intersect at 90 degrees, right angles, and hence, they are called **perpendicular**.

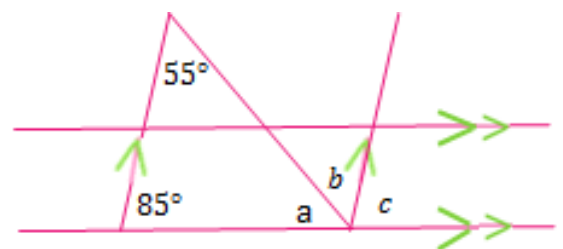


If a line crosses two other lines it is called a **transversal line** and it will create **corresponding angles**.

"Corresponding angles formed from parallel lines are equal"

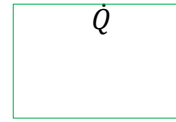
1. Your Turn:

e. Given what you have learned about parallel lines, supplementary and corresponding angles, calculate the missing angles from the image right:



2. Two Dimensional Shapes

Two dimensional shapes are also called plane shapes. One dimension is a line; only one coordinate is required to describe the position of a point Q . Whereas, with a two dimensional shape, which is a shape with two dimensions – length and width, two coordinates are required to describe the position of point Q .

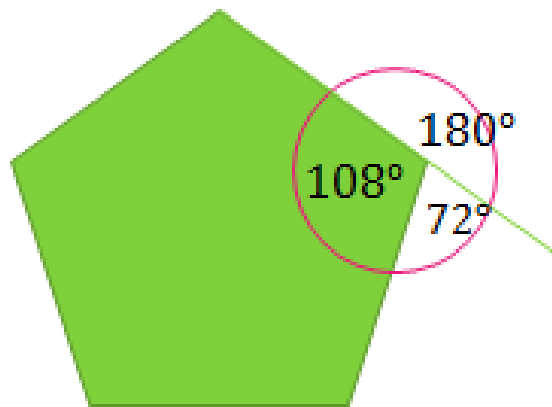


Think about describing the position of a button on a mat, two coordinates are required. Going one step further, if that button was on a three dimensional object such as a box, three coordinates would be required, length, width and height.

A two dimensional shape has **area** (the space enclosed by the bounded line segments) and a **perimeter**, the total length of the line segments.

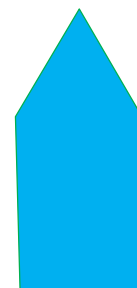
A **polygon** is a two dimensional shape that has straight sides and many angles. Poly means many and gon means angles. For instance, a pentagon, pent meaning five and gon meaning angles; and hence a pentagon has five straight lines that when enclosed make five interior angles. A **triangle** is a three sided polygon with three angles (tri-angle) and a **quadrilateral** is a four (quad) sided polygon.

A **regular polygon**: all sides are of equal length and all interior angles are equal. For example, the regular **pentagon** has each interior angle measuring 108° . Each exterior angle measures 72° and five of these make one complete turn: 360° . Also note that 108 and 72 are supplementary angles. Regular pentagons will not tessellate without the inclusion of another polygon, because to **tessellate**, the angles (vertices) need to meet at 360° ; an idea for you to investigate.



Irregular polygons do not have equivalence of angles or sides.

The shape to the right is still a five sided polygon, but it is irregular.



There are also **concave** polygons when there is an indent (cave-in) and an interior angle is *greater* than 180° .

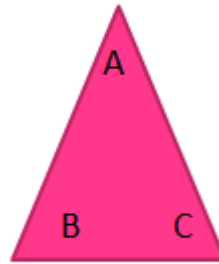
A regular polygon must be convex.



2. Your Turn:

This activity is a little different to others you have completed as you will be required to do some actual cutting and pasting. It is helpful to do this activity because it shows the how and why of mathematical ideas related to angles of triangles and quadrilaterals.

Draw a triangle and mark the angles A, B & C.
 Now cut out the triangle and tear off the corners B & C.
 Place the corner of B next to A and then corner C next to B.
 a. What do you notice?



b. What can you say about the sum of angles for a triangle?

Now draw a quadrilateral and mark the angles D, E, F & G. Cut out the quadrilateral and tear off each of the corners. Then place the corners together.



c. What do you notice?

d. What can you say about the sum of angles for a quadrilateral?

Properties of quadrilaterals:

Trapezium:

- One pair of opposite sides is parallel.

Parallelogram:

- Opposite sides are of equal length
- Diagonally opposite angles are equal
- Opposite sides are parallel
- The diagonals bisect each other (intersect at midpoint)

Rhombus:

- Diagonally opposite angles are equal
- All sides are of equal lengths
- Opposite sides are parallel
- The diagonals bisect each other at 90°

Rectangles:

- Parallelograms with all angles equal (90°)
- Opposite sides are of equal length
- The diagonals are of equal length

Kite:

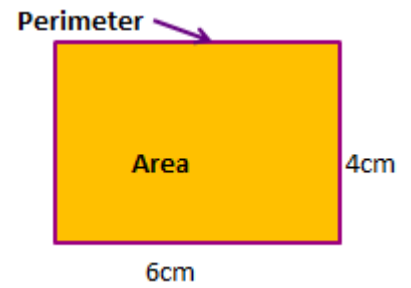
- Two pairs of sides are of equal length
- One pair of diagonally opposite angles are equal
- Only one diagonal is bisected by the other
- The diagonals cross at 90°

trapezium	
parallelogram	
rhombus	
rectangles	
kite	
arrowhead	
irregular quadrilaterals	

<http://www.lancsngfl.ac.uk/secondary/math/download/file/PDF/Shape%20Dictionary.pdf>

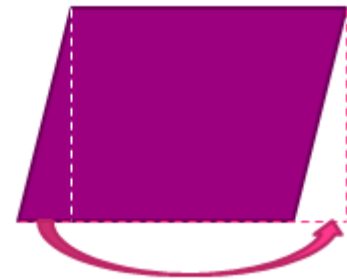
3. Area and Perimeter

Area is a measure of the amount of space a two dimensional shape takes up, that is the space that is enclosed by its boundary. The area is measured in square units and is generally calculated by multiplying the base the height: $A = b \times h$ or $A = bh$. Thus for the image right, the area would be calculated in square centimetres, so how many cm^2 fit into the enclosed space: $4cm \times 6cm = 24cm^2$



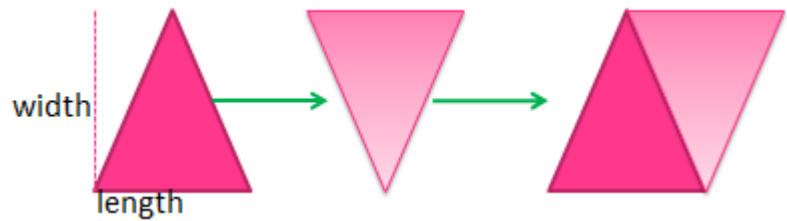
The **perimeter** is the length of the boundary. So for this image we would add the four sides: $4cm + 4cm + 6cm + 6cm = 20cm$

To calculate the area of a **parallelogram**, we still multiply the base by the height. This works because we can transpose the shape to make a rectangle: You will notice that the triangle on the right is the same as the one on the left \therefore giving the same base and height (width and length). Note: we measure the height, which distance of the line that is **perpendicular** (90°) from the base, and not along the line as such. However, the length of the line is required to accurately calculate the perimeter. Thus for this parallelogram (right) the base is 6cm and the height is 4cm; so: $A = (bh)$; $A = 6 \times 4 \therefore 24cm^2$



The same conceptual thinking applies when working with triangles. The area of a **triangle** can be calculated by $A = \frac{1}{2}(bh)$

Thus, if we take the darker triangle and rotate it, then we get a parallelogram, consisting of **congruent** (same shape and size) triangles. Hence, we calculate the area of the parallelogram as above.

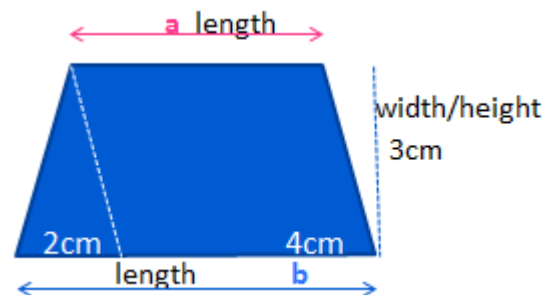


Yet, we are only interested in the triangle and so we halve the measurement (because two of the original triangles make up the area of the parallelogram). So, if the base is 3cm and the height is 4cm, we apply the formula:

$$A = \frac{1}{2}(bh); A = \frac{1}{2}(3 \times 4); \frac{3 \times 4}{2} = 6 \therefore A = 6cm^2$$


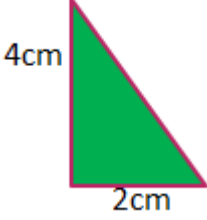
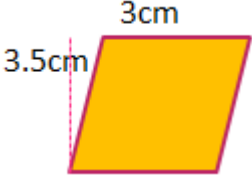
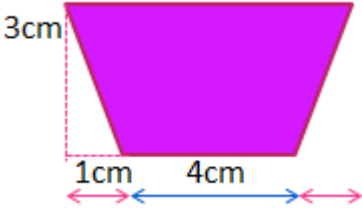
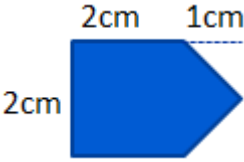
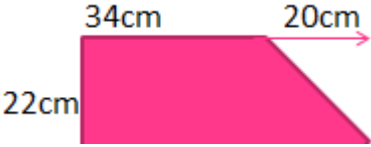
Area of a Trapezium: Given that a trapezium has one set of opposite sides that are parallel, we can apply the above concepts to calculate its area. This principle applies because a trapezium can be transformed into a triangle and a parallelogram as shown. Hence, the area would be area of the rectangle + area of the triangle. Mathematically we can simplify further:

$$A = \frac{1}{2}(a + b) \times h \text{ thus: } A = \frac{1}{2}(4 + 6) \times 3 = 15cm^2$$



3. Your Turn:

Calculate the area for the following:

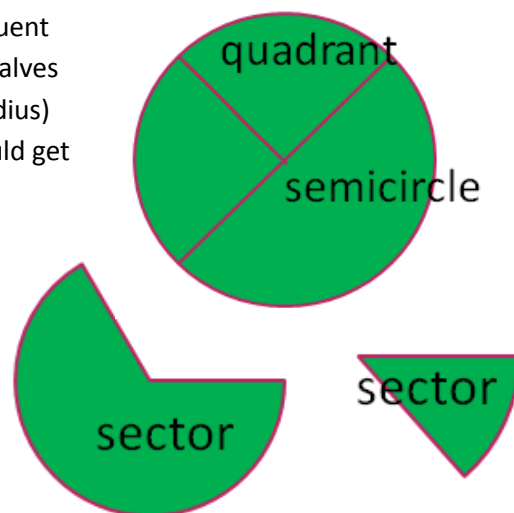
a. 	b. 	c. 
$A =$	$A =$	$A =$
d. 	e. 	f. 
$A =$	$A =$	$A =$

4. Circles

Circles: A circle is an enclosed shape where all points on the perimeter are at a *fixed* distance from the **centre**. The perimeter of a circle is called the **circumference**. The line from the centre to any given point on the circumference is the **radius**, and the line from one point, through the centre to another point on the circumference, is the **diameter**.

The diameter will divide the circle into two congruent halves. Congruent meaning the halves will be exactly the same size and shape. These halves are called **semi-circles**. Then if we were to draw a line segment (radius) that was perpendicular to the diameter in a semicircle, then we would get two congruent **quadrants**.

Any two radii (plural for radius) of a given circle will create a **sector**.



Circumference of a Circle



The diameter of a circle (the broken line) will fit around the circumference (the solid blue line) of a circle three and a bit times, as shown. If the diameter of a circle is doubled, then so will the circumference of a circle. Hence, we could say that the diameter is proportional to the circumference. Hence, a ratio exists between the diameter and the circumference of a circle. The ratio is 1: 3.14159265358979. This can be rounded to two decimal places, hence, the ratio is expressed as 1:3.14. The 3.14 (three and a bit) is represented symbolically as π . The symbol is called Pi; it denotes the special ratio: $\frac{\text{circumference}}{\text{diameter}} = \pi$. Pi is irrational, however, $\frac{22}{7}$ is a rational approximation often used. In short, this ratio denotes that the circumference of a circle is 3.14 times bigger than its diameter.

Mathematically the formula to calculate the **circumference of a circle** is $c = \pi d$. Thus if the diameter of a circle is 5cm , then the circumference can be calculated: $c = \pi d$; $c = 3.14 \times 5 = 15.7 \therefore c = 15.7\text{cm}$

Note that the answers most often will be approximate because we often use an approximation for π

4. Your Turn:

Find the circumference of a circle for each of the following giving the answer: i) using an approximate value for $\pi \approx 3.14$ or ii) as an approximate value $\pi \approx \frac{22}{7}$ (using the rational number $\frac{22}{7}$)

a. radius: 7cm

i)

ii)

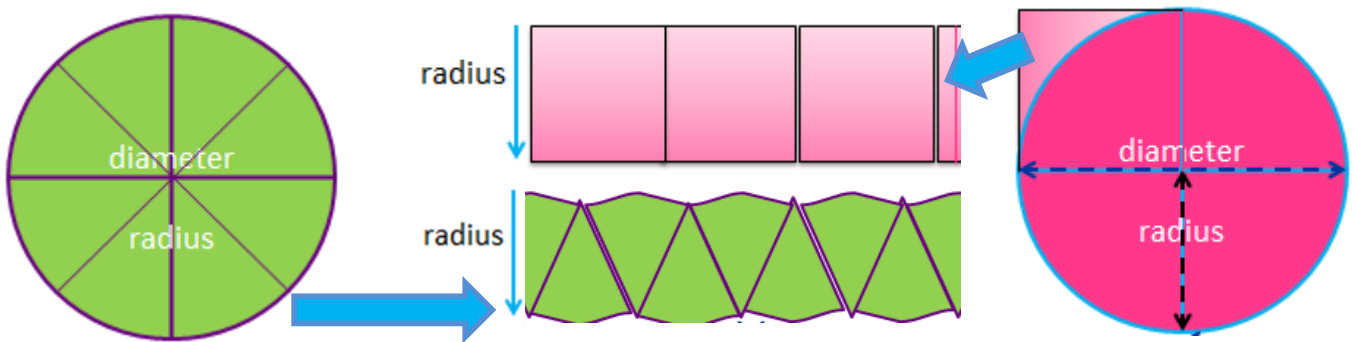
b. diameter: 42cm

i)

ii)

Area of a circle

The **area of a circle** is calculated by the formula $A = \pi r^2$. This formula says that if we multiply $\pi(3.14)$ by the square of the radius, then it will give us the space taken up by the circle. Let's investigate:



To illustrate how the formula works, the green circle above has been cut into segments to be transformed into a rectangular shape. Each segment has been placed in a top and tail formation creating a different shape. Do this for yourself: draw a circle and cut it into segments to make the same configuration as above. This shape represents the surface area of the circle in another way, a rectangular shape. The radius is the width as shown, and the length represents a close approximation of the circumference of the circle.

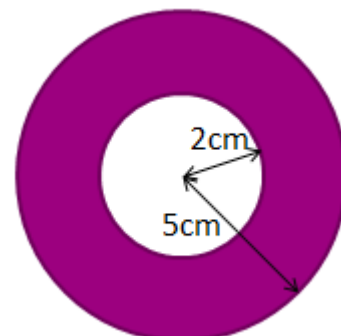
Now examine the pink circle. The square of the radius has been marked with pale pink. Next to that is three and a bit (π which is 3.14) 'radius squares'. When we compare the two diagrams you will note that the pale pink and the green rectangular shapes are very similar in size. You can do this too. Take the same circle used previously, mark and cut out the radius square. Now see how many times it will fit across your rectangular (circle) shape.

4. Your Turn:

Given the circle to the right, answer the following:

c. What is circumference of the outer circle?

d. What is the area of the purple region (the **annulus**)?



5. Three Dimensional Shapes

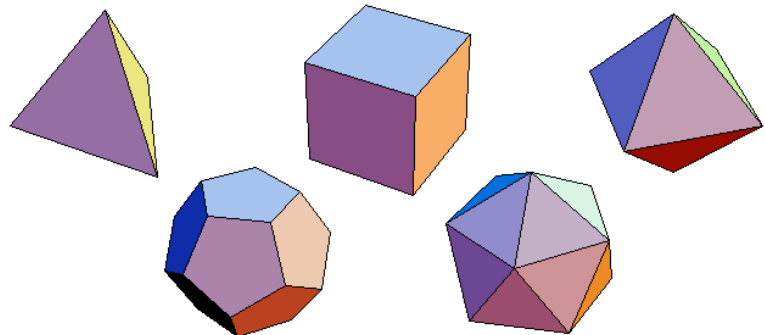
We are surrounded by three dimensional shapes in our daily lives. Three dimensional shapes have length, breadth/width and height. They also have faces, edges and vertices (vertex for singular).



- Faces: plane surfaces (not sides)
 - A sphere has one continuous smooth surface – it is not called a face
- Edges: the lines where two faces meet
- Vertices: the points where the edges meet

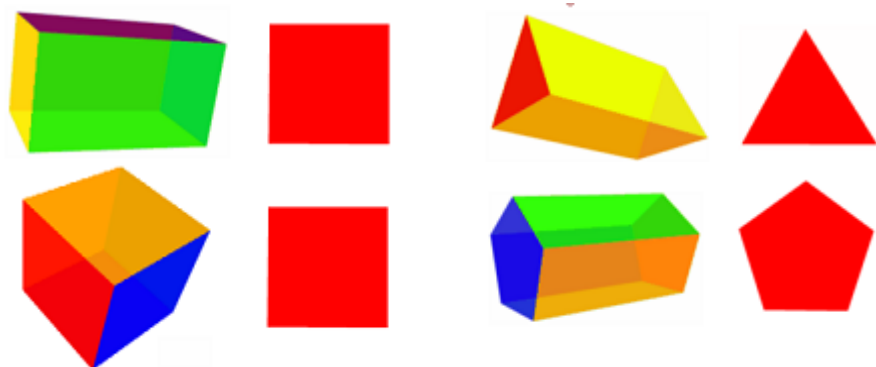
Three dimensional shapes are also called **solids**. A solid figure can also be called a **polyhedron**, which means that the faces are polygons. There are only five kinds of regular polyhedra:

- Regular tetrahedron: 4 faces, each an equilateral triangle
- Cube (regular hexahedron)
- Regular octahedron: 8 faces, each an equilateral triangle
- Regular dodecahedron: 12 faces, each a regular pentagon
- Regular icosahedron: 20 faces, each an equilateral triangle



<https://sunnemath.wordpress.com/tag/geometry-2/>

Prism: a prism has congruent and parallel bases (top and bottom) and are joined together by lateral rectangular faces, in other words, a solid consisting of two identical polygons joined at either end by parallel lines (rectangles).



A rectangular prism is called a cuboid.
A cube is a special kind of cuboid

Also displayed (right) is a triangular prism and a pentagonal prism – prisms are named after the base face and have a **uniform cross-section**.

<http://www.mathsisfun.com/geometry/prisms.html>

Three dimensional shapes have surface area and volume or capacity:

- Volume: the amount of 3D space occupied
 - Liquid volume: Units measured in litres (L) and millilitres (mL)
 - Solid volume: Unit measures include cubic centimetres (cm^3) and cubic metres (m^3).
- Capacity: This is a term to describe how much a container will hold; only containers have capacity.
 - Units of capacity (volumes of fluids or gases) are also measured in litres (L) and millilitres (mL).

The volume of a **rectangular prism** is the *length* by the *breadth/width* by the *height*: $V = lbh$

The volume of prisms involves calculating the **surface area** of the *polygon base* by the *height*: $V = Ah$

EXAMPLE:

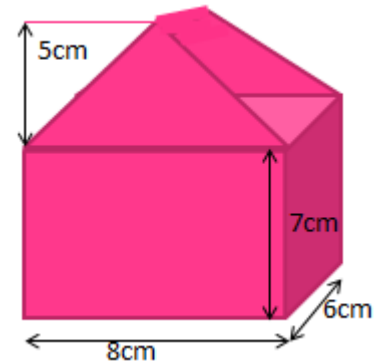
Find the volume of the prism shown:

First, we calculate the area of the front face, the cross section, that is then multiplied by the height for $V = Ah$. As you can see the front face consists of a triangle and a rectangle. Hence, first two formulas are applied.

$$A = \frac{1}{2}(lb) + (lb) \text{ (add the areas of a triangle and rectangle)}$$

$$\text{So, } A = \frac{1}{2}(5 \times 8) + (7 \times 8); A = 20 + 56 = 76 \therefore A = 76cm^2$$

Now we multiply that by the height (depth) ($6cm$), let's assume the shape is on its side: $V = Ah; V = 76 \times 6 = 456 \therefore V = 456cm^3$



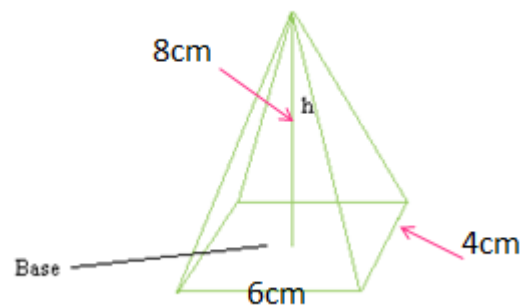
The volume of a pyramid and a cone

The volume of a pyramid is a third of the volume of a rectangular prism with the same height and the same rectangular base. If the base of both shapes were open and we filled them with water, the prism will take three times the amount of water as the pyramid, hence a ratio of: *volume of prism : volume of pyramid* is 3: 1

To calculate the volume of a pyramid (right) we use the formula:

$$V = \frac{1}{3}Ah \text{ Hence, the area of the is } 24cm^2$$

$$\text{Thus: } V = \frac{1}{3}24 \times 8 \therefore V = 64cm^3$$



Adapted from <http://www.basic-mathematics.com>

Let's check the theory that the volume of a pyramid is one third of the volume of a rectangular prism of the same height and base. $V = lbh; V = 6 \times 4 \times 8 = 192cm^3$; so to check $\frac{192}{3} = 64$ ✓

The same applies for a cone and a cylinder, the ratio of: *volume of cylinder : volume of cone* is 3: 1

5. Your Turn:

a. Calculate the volume of a cylinder with the radius of $4cm$ and height of $6cm$.



b. Calculate what the volume of the cone would be if it had a base with a $4cm$ radius and the height of $6cm$.

6. Measurement

Measuring and calculating areas and volumes is a skill used in our daily lives. For instance, we might need to know how much water will be required to fill the pool, or how much carpet to buy for the bedroom, or how to alter the quantities of ingredients if the radius of a cooking tin doubles.

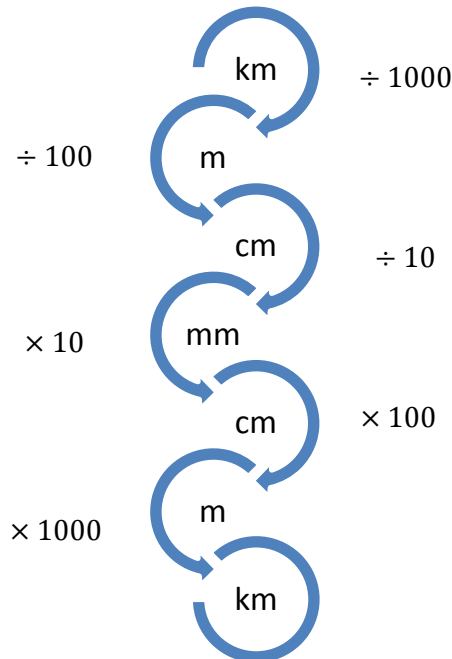
In Australia the metric system of measurement is used. This system is base 10 which does make calculations simpler than the base sixty system used for time. Many charts are available online to show units of measure. These charts are also found in diaries like the one to the right.

Length	Weight	Volume
1 km = 1,000 m	1 kg = 1,000 g	1 kL = 1,000 L
1 m = .001 km	1 g = .001 kg	1 L = .001 kL
1 m = 100 cm	1 g = 100 cg	1 L = 100 cL
1 cm = .01 m	1 cg = .01 g	1 cL = .01 L
1 m = 1,000 mm	1 g = 1,000 mg	1 L = 1,000 mL
1 mm = .001 m	1 mg = .001 g	1 mL = .001 L

It is also helpful to know some of the more common measurement prefixes:

Prefix	Symbol	Multiplying Factor
Mega-	M	One million = 10^6
Kilo-	k	One thousand = 10^3
Deci-	d	One tenth = $\frac{1}{10}$
Centi-	c	One hundredth = $\frac{1}{10^2}$
Milli-	m	One thousandth = $\frac{1}{10^3}$
Micro-	μ	One tenth = $\frac{1}{10^6}$

When converting units of measure we multiply or divide by a base of ten, for example:



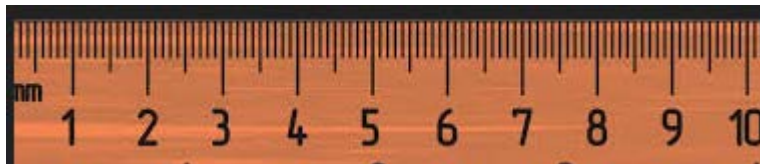
Unit conversion

There are various types of measurements with various units of measure. For instance, we measure time, distance, speed, weight and so on. Then time is measured in seconds, minutes, hours, weeks, days, months, years...

Often we are required to convert a unit of measure; we may be travelling and need to convert measurements from imperial to metric, or we may need to convert millimetres to metres for ease of comparison. In the fields of science and medicine, converting measurement can be a daily activity.

It helps to apply a formula to convert measurement. However, it is essential to understand the how and why of the formula otherwise the activity becomes one we commit to memory without understanding what is really happening. More mistakes are made when procedures are carried out without understanding, which could be particularly troublesome in the field of medicine.

The metric system is base 10 which makes unit conversion relatively easy. For example, we know that there are 10mm in a cm, and there are 100cm in a metre. The metre is a standard unit of **length** and it helps to understand what a metre looks like



so as you get a 'feel' for measurement. The height of your kitchen bench might be approximately a metre, and the width of your little finger might be equivalent to a centimetre.

Let's look at conversions symbolically.

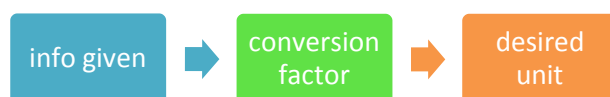
$$1m = 100cm = 1000mm \text{ or we could say that } 1mm = \frac{1}{10}cm = \frac{1}{1000}m$$

Unit Conversion rules:

- Always write the unit of measure associated with every number.
- Always include the units of measure in the calculations.

Unit conversion requires algebraic thinking which will be covered in the next booklet; however, here you will be introduced to the concept. Let's convert 58mm into metres. $58mm \times \frac{1m}{1000mm} = 0.058m$

The quantity $\frac{1m}{1000mm}$ is called a **conversion factor**; it is a division/quotient; in this case it has metres on top and mm on the bottom. Another name for the conversion factor is a solution map. So we can see that we work with information given, a conversion factor, and then the desired unit.



EXAMPLE PROBLEM:

Convert 32centimetres to metres. There are 100cm in a metre so our solution map is $\frac{1m}{100cm}$

$$\text{The working is as follows: } 32cm \times \frac{1m}{100cm} = 0.32m$$

Here we can cancel out the cm and are left with metres and the new just need to divide 32 by 100 to get 0.32m.

It is helpful to have a thinking process to follow. This one comes from the book, *Introductory Chemistry* (Tro, 2011, pp. 25-35). There are four steps to the process: sort, strategise, solve and check.

- **Sort:** First we sort out what given information is available.
- **Strategise:** The second step is where a solution map is created. You may need to look at a conversion table and possibly use a combination of solution maps.
- **Solve:** This is the third step which simply means to solve the problem.
- **Check:** The last step requires some logical thought; does the answer make sense?

EXAMPLE PROBLEM:

Convert 2 kilometres (km) into centimetres (cm).

- **Sort:** we know there are 1000 metres in one km, and 100cm in one metre.
- **Strategise:** So our maps could be $\frac{1000m}{1km}$ and $\frac{100cm}{1m}$
- **Solve:** $2km \times \frac{1000m}{1km} \times \frac{100cm}{1m} = x \text{ cm}$
 $2km \times \frac{1000m}{1km} \times \frac{100cm}{1m} = 2 \times 1000 \times 100 \text{ cm} \therefore 2km = 200,000cm$
- **Check:** is there 200,000cm in a kilometre? Yes that seems sensible.

6. Your Turn:

Convert the following:

- | | |
|-------------------------|-------------------------|
| a. 285m into kilometres | b. 96cm into kilometres |
| ◦ Sort | ◦ Sort |
| ◦ Strategise | ◦ Strategise |
| ◦ Solve | ◦ Solve |
| ◦ Check | ◦ Check |

c. Using this information: $1 \text{ m}^2 = 10000\text{cm}^2 = 1000000\text{mm}^2$

Convert 1.5m^2 into mm^2

- Sort
- Strategise
- Solve
- Check

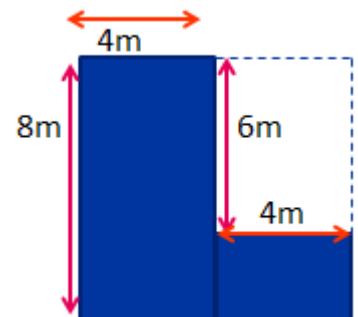
$1m = 100cm = 1000m$



7. Area and Volume (extension)

Sometimes an area may consist of composite shapes/figures. For example, we may want to carpet the floor of a room that is L-shaped. To calculate the area we may need to add or subtract. For the figure (right) we could approach it a few ways, we could add the two rectangles $8 \times 4 = 32\text{m}^2$ and $4 \times 2 = 8\text{m}^2$ to get 40m^2

Or we could subtract the rectangle that is the blank space: $6 \times 4 = 24\text{m}^2$ from the square that could be made from the space: $8 \times 8 = 64\text{m}^2$ to also end up with 40m^2

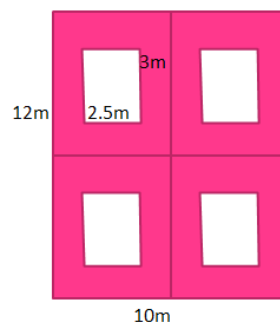


7. Your Turn:

a. What is the area of the shaded region?

Another example to extend your thinking:

b. If you have a recipe for a cake that is to be cooked in a tin that has a diameter of 20cm and is 10cm high, yet your tin has a diameter 40cm and is 10cm high, then do you simply double the recipe?



8. Answers

- a. $40^\circ + 140^\circ = 180^\circ$; $\angle AOL$ 40° and $\angle AOB$ 140° b. $45^\circ + 45^\circ = 90^\circ$ and $25^\circ + 65^\circ = 90^\circ$
c. $40^\circ + 140^\circ + 40^\circ + 140^\circ = 360^\circ$ d. 'Vertically opposite angles are equal'
e. $a=40^\circ$ $b=55^\circ$ $c=85^\circ$
- a. They meet up on a straight line b. The sum of angles for a triangle is 180°
c. They make a full circle d. The sum of angles for a quadrilateral is 360°
- a. 4cm^2 b. 4cm^2 c. 10.5cm^2 d. 15cm^2 e. 5cm^2 f. 968cm^2
- a. i) $c = \pi d$ so $c \approx 3.14 \times 14 = 43.96\text{cm}$ ii) $c \approx \frac{22}{7} \times 14 = 44\text{cm}$
b. i) $c = \pi d$ so $c \approx 3.14 \times 42 = 131.88\text{cm}$ ii) $c \approx \frac{22}{7} \times 42 = 132\text{cm}$
c. $c = \pi d$ so $c \approx 3.14 \times 10 = 31.4\text{cm}$
d. $A = \pi r^2$; Large circle: $3.14 \times 25 = 78.5\text{cm}^2$; less small circle $3.14 \times 4 = 12.56\text{cm}^2$
 \therefore The area of the purple region is 65.94cm^2
- a. Cylinder Volume: $V = \pi r^2 h$; Cylinder base area: 50.24cm^2 $V = 301.44\text{cm}^3$
b. Cone Volume: $V = \frac{1}{3} \pi r^2 h = 100.5\text{cm}^3$
- a. 0.285km b. 0.00096km or $9.6\text{km} \times 10^{-4}$ c. 1500000mm^2 or $1.5\text{mm}^2 \times 10^6$
- a. 90cm^2
b. The recipe will need to be multiplied by four. Tip: calculate the volume of both tins

9. Helpful Websites

Properties of 2D Shapes: <http://www.lancsngfl.ac.uk/secondary/math/download/file/PDF/Shape%20Dictionary.pdf>

Area and Perimeter: http://www.amsi.org.au/teacher_modules/Parallelograms_and_rectangles.html

Properties of 3D Shapes: <http://www.mathsisfun.com/geometry/prisms.html>

Units of Measure: <http://www.staff.vu.edu.au/mcaonline/units/measure/meauni.html>