Maths Refresher

Roots and Powers

Learning, Teaching
and Student Engagement

Roots and Powers

Learning intentions ....

- Powers
- Scientific notation
- Indices
- Index laws
- Square roots
- Cube roots
- Root operations
- Logarithms

Powers

- "Powers" are a method of simplifying equations.

- If I had a sum: $7 + 7 + 7 + 7 + 7 + 7 + 7 + 7 + 7 = ?$
  I could simplify by writing $7 \times 9 = ?$

- If I had a multiplication:
  - $7 \times 7 \times 7 \times 7 \times 7 \times 7 \times 7 \times 7 \times 7 = ?$
    I could simplify by writing $7^9 = ?$

- A simple way to describe powers is to think of them as how many times you multiply the **base** number by itself.

Powers

5^4

The 4 is called either the **exponent** or **index**.

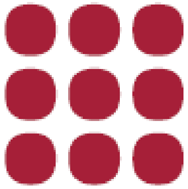
The 5 is called the **base**.

- The expression 5^4 is called a **power** of 5
- The raised 4 in 5^4 is called the **index** or **exponent** or the **power**
- The number 5 in 5^4 is called the **base** of the power
- The exponent is written as a superscript.
- Positive exponents indicate the number of times a term is to be multiplied by itself.

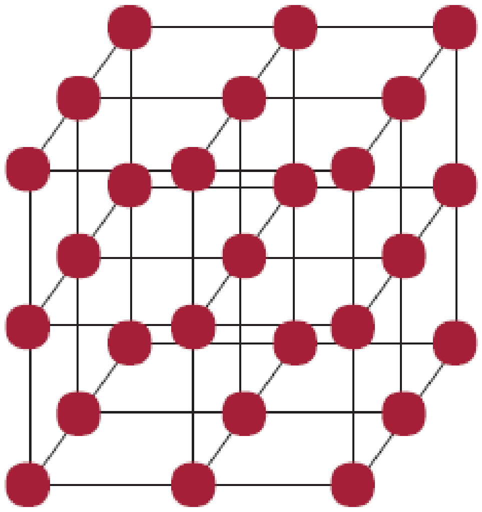
5^4 is the same as $5 \times 5 \times 5 \times 5$, which is the same as 625
The most common way to describe the number is to call it "five to the power of four"

Powers

- **For example:**
- The two most common powers (2 & 3) are given names related to geometry, as shown.
  For example:
  - 3^2 is called "three squared" and
  - 3^3 is called "three cubed"



3^2



3^3

Powers

The powers of 2 are: 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 4096, 8192,...

The powers of 3 are: 3, 9, 27, 81, 243, 729,...

The powers of 4 are every second power of 2.

The powers of 5 are: 5, 25, 125, 625, 3125,...

The powers of 6 are: 6, 36, 216,...

The powers of 7 are: 7, 49, 343,...

The powers of 8 are every third power of 2.

The powers of 9 are every second power of 3.

The powers of 10 are: 10, 100, 1000, 10 000, 100 000, 1 000 000,...

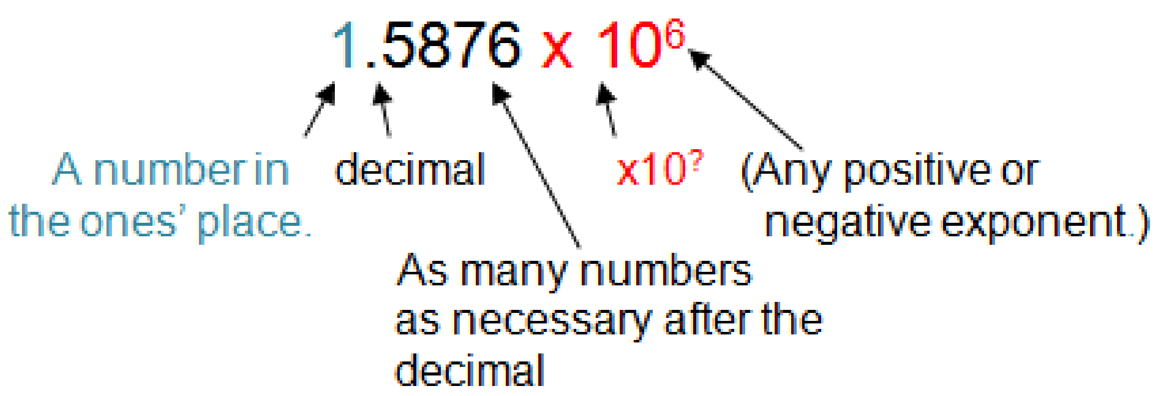
The powers of 16 are every fourth power of 2.

It helps to be able to recognise some powers for later when working with logarithms

Scientific notation

- As we explored in week one, our place value system of 10 displays every number as a product of multiples of powers of 10.

Scientific notation must always be written with the same components as the following model:

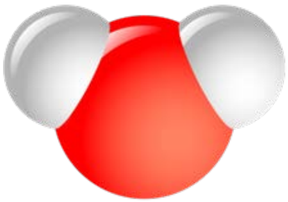


Scientific Notation

Scientific notation relates to place value.

For example:

- $10^5 = 10 \times 10 \times 10 \times 10 \times 10 = 100\ 000$. (Note: 100 000 has 5 places to the right from one)
- $10^{-4} = \frac{1}{10^4} = 10 \div 10 \div 10 \div 10 = 0.0001$. (Note: 0.0001 has 4 places to the left from one)
- $3000 = 3 \times 1000 = 3 \times 10^3$ (3places to the right)
- $704\ 500\ 000 = 7.045 \times 100\ 000\ 000 = 7.045 \times 10^8$ (8 places to the right)
- The number of molecules in 18 grams of water is 602 000 000 000 000 000 000 000 which is written as 6.02×10^{23}



KHANACADEMY

Watch this short Khan Academy video for further explanation:
"Scientific notation"
https://www.khanacademy.org/math/algebra-basics/core-algebra-foundations/algebra-foundations-scientific-notation/v/scientific-notation

Your turn….

SCIENTIFIC NOTATION

Example problems:

1. $3459 = 3.459 \times 10^3$
2. $0.000004567 = 4.567 \times 10^{-6}$

Practise problems:

Write the following in scientific notation:

1. 450
2. 90000000
3. 3.5
4. 0.0975

Write the following numbers out in full:

1. 3.75×10^2
2. 3.97×10^1
3. 1.875×10^{-1}
4. -8.75×10^{-3}

Answers

Write the following in scientific notation:

450 = 4.5×10^{2}

90000000 = 9.0×10^{7}

3.5 = is already in standard form

0.0975 = 9.75×10^{-2}

Write the following numbers out in full:

3.75×10^{2} = 375

3.97×10^{1} = 39.7

1.875×10^{-1} = 0.1875

-8.75×10^{-3} = -0.00875

Indices

- Working with powers/indices assist calculations and simplifying problems.
- There are laws which assist us to work with indices.
- The next few slides will explain the ‘index laws’:

The first law

- $a^m \times a^n = a^{m+n}$
- How does this work?
- Let's write out the 'terms'
- $a^7 \times a^2 = a^{7+2} = a^9$

$$\overbrace{(a \times a \times a \times a \times a \times a \times a)}^{7} + \overbrace{(a \times a)}^{2}$$

Watch this short Khan Academy video for further explanation:
"Simplifying expressions with exponents"
https://www.khanacademy.org/math/algebra/exponent-equations/exponent-properties-algebra/v/simplifying-expressions-with-exponents

The second law

- $\frac{x^9}{x^6} = x^{9-6} = x^3$
- How does this work?
- $\frac{x \times x \times x \times x \times x \times x \times x \times x \times x}{x \times x \times x \times x \times x \times x}$ = apply cancel method
- $\frac{x \times x \times x \times \cancel{x} \times \cancel{x} \times \cancel{x} \times \cancel{x} \times \cancel{x} \times \cancel{x}}{\cancel{x} \times \cancel{x} \times \cancel{x} \times \cancel{x} \times \cancel{x} \times \cancel{x}} = \frac{x \times x \times x}{1}$
- $= x \times x \times x = x^3$
- From the second law we learn why x^0=1
- Any expression divided by itself equals 1
- so $\frac{x^3}{x^3} = 1$ or x^{3-3}=x^0 which is 1

The third law

- $(b^a)^m = b^{am}$
- $(b^2)^3$
- How does this work?
- $(b \times b) \times (b \times b) \times (b \times b) = b^6$
- Therefore, we multiply the indices.

Index laws

First law	$a^m \times a^n = a^{m+n}$
Second law	$\frac{a^m}{a^n} = a^{m-n}$
Third law	$(a^m)^n = (a^n)^m = a^{nm}$

And

IL4	$a^0 = 1 \ \ (a \neq 0)$
IL5	$a^1 = a$
IL6	$a^{-m} = \frac{1}{a^m} \ (a \neq 0)$ $a^m = \frac{1}{a^{-m}}$
IL7	$a^{1/m} = \sqrt[m]{a}$

IL8	$a^m b^m = (ab)^m$

Q: What do you notice about these laws?

A: In each case, there is only one value for the base!

There are other Index Laws that help you deal with problems where the bases are different, and/or where the indices are different. For example, in *IL8*, there are two different base values, and only one index value.

Your turn …

THE FIRST INDEX LAW

Simplify

$a^4 b^2 b^3 a^6$

The Second Index Law

Simplify

$$\frac{z^5}{z^3}$$

The Third index law

$(b^9)^2$

Working with index laws

We often need to use several laws of indices to solve one problem. For example:

Simplify $\frac{(x^3)^4}{x^2}$

From the third law: $(x^3)^4 = x^{12}$

So, $\frac{x^{12}}{x^2} = x^{10}$ (from the second law)

Working with the index laws

Powers can be simplified if they are **multiplied** or **divided** and have the **same base**.

Problem	Simplified	Law
5^1	5	
4^0	1	
2^3 x 2^2	2^3 x 2^2 = 2^{3+2} = 2^5	
$2^3 \div 2^2$	$2^{3-2} = 2^1 = 2$	
$(2^3)^2$	$2^{3 \times 2} = 2^6$	

Your turn …

First law	$a^m \times a^n = a^{m+n}$
Second law	$\frac{a^m}{a^n} = a^{m-n}$
Third law	$(a^m)^n = (a^n)^m = a^{nm}$

IL4	$a^0 = 1 \quad (a \neq 0)$
IL5	$a^1 = a$
IL6	$a^{-m} = \frac{1}{a^m} \quad (a \neq 0)$ $a^m = \frac{1}{a^{-m}}$
IL7	$a^{1/m} = \sqrt[m]{a}$

Practise problems:

1. Simplify $5^2 \times 5^4$
2. Simplify $X^2 \div X^5$
3. Evaluate 14^0
4. Evaluate 5^1
5. Simplify $(5^4)^3$
6. Simplify $x^{1/2}$

Answers

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WORKING WITH INDEX LAWS

$5^2 \times 5^4 = 5^6$

$X^2 \div X^5 = X^{2-5} = X^{-3}$

$14^0 = 1$

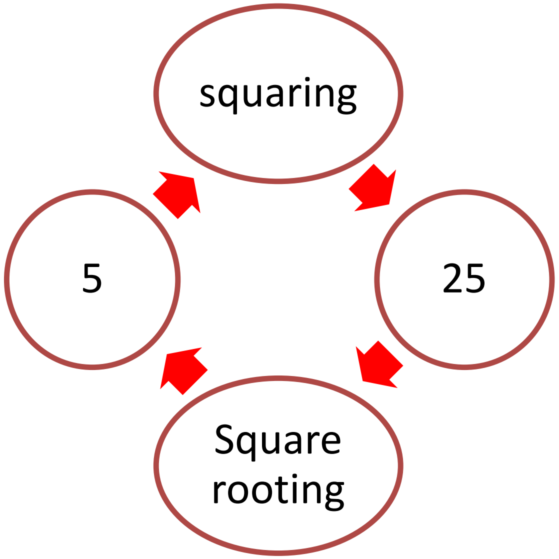
$5^1 = 5$

$(5^4)^3 = 5^{4 \times 3} = 5^{12}$

$x^{½} = \sqrt[2]{x}$

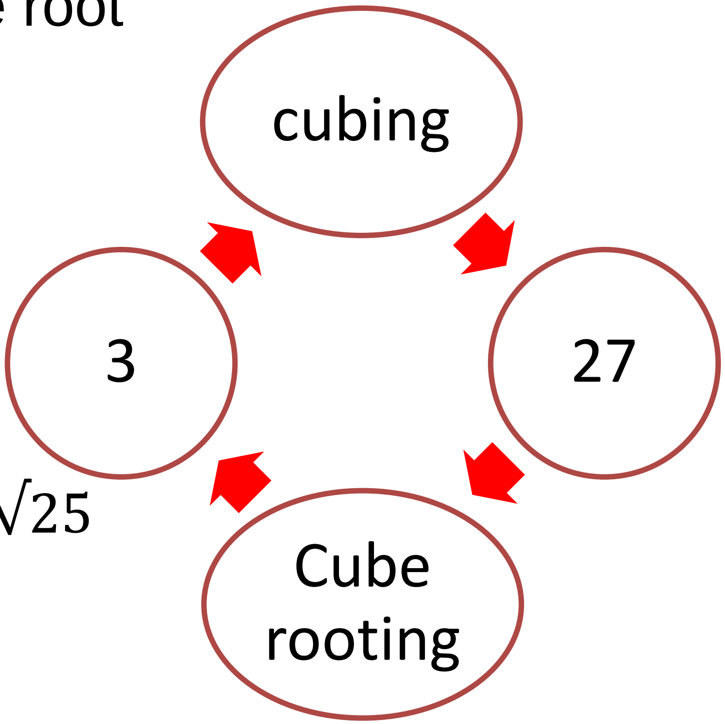
Square Roots

- If we look at the relationship between 5 and 25
  - What can we say about the numbers?
  - We know that 5 squared is 25 (5x5)
  - So we could then say that 25 is the square of five – which means that 5 is the square root of 25
  - $\sqrt{25} = 5$ *and* 5^2=25



Cube Roots

- If we look at the relationship between 3 and 27
  - What can we say about the numbers?
  - We know that 3 cubed is 27 ($3 \times 3 \times 3$)
  - So we could then say 3 is the cube root of 27
  - $\sqrt[3]{27} = 3 \; and \; 3^3$=27
- Also when working with negative numbers
  - $(-2)^3$ = -8
  - $-5 \; is \; also \; the \; square \; root \; of \; 25$
  - $(-5) \times (-5)$ = 25 so -5 is also the $\sqrt{25}$



More on Roots

- A root is used to find an unknown base.

$$\sqrt{a}$$

- Here, the root symbol is called the ***radical symbol***, and the *a* is referred to as the ***radicand***.

- Like indices, the two most common roots (2 & 3) are called square root and cube root.

  For example:

  $\sqrt{64}$ is called “the square root of 64”.

  $\sqrt[3]{27}$ is called “the cube root of 27”

  (*Note*: square root does not have a 2 at the front, it is assumed).

- In words, $\sqrt{64}$ means “what number multiplied by itself equals 64”. What do you think $\sqrt[3]{27}$ means in words?

More on Roots

- Simplifying roots is difficult and, as such, being able to estimate the root of a number is a useful practice – particularly when you don’t have a calculator.

- To estimate a root, we must know the common powers. Some common powers are given in the table on the next slide:

Power		Answer
1^2	$(-1)^2$	1
2^2	$(-2)^2$	4
3^2	$(-3)^2$	9
4^2	$(-4)^2$	16
5^2	$(-5)^2$	25
6^2	$(-6)^2$	36
7^2	$(-7)^2$	49
8^2	$(-8)^2$	64
9^2	$(-9)^2$	81
10^2	$(-10)^2$	100
11^2	$(-11)^2$	121
12^2	$(-12)^2$	144

Using the table:

$\sqrt{81} = \pm 9$

So, there are two square roots of any positive number.

What about negative numbers?
For example, does -100 have any square roots?

No, it doesn't.
Why not?

Negative numbers don't have square roots because a square is either positive or zero

Power		Answer
1^2	$(-1)^2$	1
2^2	$(-2)^2$	4
3^2	$(-3)^2$	9
4^2	$(-4)^2$	16
5^2	$(-5)^2$	25
6^2	$(-6)^2$	36
7^2	$(-7)^2$	49
8^2	$(-8)^2$	64
9^2	$(-9)^2$	81
10^2	$(-10)^2$	100
11^2	$(-11)^2$	121
12^2	$(-12)^2$	144

But what about if the radicand is not a perfect square?

For example:

$$\sqrt{56} = ?$$

Using the table, we can estimate that the answer is a number between 7 and 8 (actual answer ± 7.48).

A ***surd*** is a special root which cannot be simplified into a whole number.

For example, $\sqrt{4}$ =2

2 is a whole number, therefore $\sqrt{4}$ is not a surd.

In contrast,

$\sqrt{3}$ =1.732

1.732 is not a whole number, therefore $\sqrt{3}$ is a surd.

Large roots e.g. $\sqrt{56}$ must be simplified to determine if they are surds. This process is explained on the next slide.

Root Operations

Simplify $\sqrt{\mathbf{56}}$

56 has multiple factors: 1×56 or 2×28 or 4×14 or 7×8.

4×14 are the key factors, since one of them (4) is a square number.

So, we can simplify $\sqrt{56} = \sqrt{4} \times \sqrt{14} = 2\sqrt{14}$

(because $\sqrt{4} = 2$)

KHANACADEMY

Watch this short Khan Academy video for further explanation:
"Simplifying square roots"
https://www.khanacademy.org/math/algebra-basics/core-algebra-foundations/square-roots-for-college/v/simplifying-square-roots-1

Root operations

Rules	Example
$\sqrt{a}\ \sqrt{b} = \sqrt{ab}$	$\sqrt{6} \times \sqrt{4} = \sqrt{6 \times 4} = \sqrt{24}$
$\sqrt{a} \div \sqrt{b} = \frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$	$\sqrt{25} \div \sqrt{16} = \frac{\sqrt{25}}{\sqrt{16}} = \sqrt{\frac{25}{16}}$
$\sqrt{a^2} = \lvert a \rvert$	$\sqrt{5^2} = 5$ (absolute value)

Your turn …

Working with root operations

1. Indicate which of these is the radical and which is the radicand $\sqrt{a}$
2. What are the square roots of the following numbers:

   100, 64, 9
3. Simplify $\sqrt{4}\ \sqrt{16}$
4. Simplify the square root of 54.
5. Use your calculator to find the cube root of 37. Is this a surd?

Answers

1. Indicate which of these is the radical and which is the radicand $\sqrt{a}$
2. What are the square roots of the following numbers:
   100 (± 10); 64 (± 8); 9 (± 3)
3. Simplify $\sqrt{4}\ \sqrt{16} = \sqrt{4 \times 16} = \sqrt{64} = 8$
4. Simplify the square root of 54 = $\sqrt{9} \times \sqrt{6} = 3\sqrt{6}$
5. Use your calculator to find the cube root of 37 = 3.332

   Is this a surd? Yes

Your turn …

DIRECTIONS: Find each square root.

1. $\sqrt{9}$
2. $\sqrt{36}$
3. $\sqrt{100}$
4. $\sqrt{81}$
5. $\sqrt{1}$
6. $\sqrt{4}$
7. $-\sqrt{25}$
8. $\sqrt{36}-\sqrt{49}$
9. $\sqrt{121}$
10. $\sqrt{64}+\sqrt{4}$
11. $-\sqrt{36}+\sqrt{9}$
12. $\sqrt{49}-\sqrt{25}$

Answers

1. ± 3
2. ± 6
3. ± 10
4. ± 9
5. ± 1
6. ± 2
7. -5
8. -1
9. ± 11
10. 10
11. -3
12. 2

Logarithms

- **Logarithms count multiplication as steps**
- Logarithms describe changes in terms of multiplication: in a log10 problem, each step is $10 \times$ bigger. With the natural log, each step is “e” (2.71828…) times more.
- When dealing with a series of multiplications, logarithms help “count” them
- For example:

$1000 = 10 \times 10 \times 10 = 10^3$, the index 3 shows us that that there have been 3 lots of multiplication by 10 to get from 10 to 1000.

Logarithms

- Given an equation such as $125 = 5^3$, we call 5 the base and 3 the exponent or index.
- We can use logarithms to write the equation in another form. The logarithm form is

  $\log_5 125 = 3$
- This is read as “logarithm to the base 5 of 125 is 3”.
- In general, if $y = a^x$

  then

  $\log_a y = x$

In other words:
$y = a^x$ and $\log_a y = x$ are equivalent

Logarithms

Worked example.

Write the following in logarithmic form:

$16 = 4^2$

$2 = \log_4 16$

In words: 2 is the logarithm to base 4 of 16.

Let's try another:

$8 = 2^3$

$3 = \log_2 8$

In words:
3 is the logarithm to base 2 of 8

Logarithms

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Logarithms can also be written in exponential form. For example:

- $\log_2 16 = 4$

Here, the base is 2, so we can write

$16 = 2^4$

Let's try another one together:

$\log_3 27 = 3$

Here, 3 is the base, and so

$27 = 3^3$

There are laws of logarithms that should be followed when working with logarithms

Logarithms

- Khan Academy video **“Logarithms”** https://www.khanacademy.org/math/algebra2/logarithms-tutorial/logarithm_basics/v/logarithms

- Using logs in the real world
  http://betterexplained.com/articles/using-logs-in-the-real-world/

- TED talk: Logarithms explained
  http://ed.ted.com/lessons/steve-kelly-logarithms-explained

Roots and Powers

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Reflect on the learning intentions ….

- Powers
- Scientific notation
- Indices
- Index laws
- Square roots
- Cube roots
- Root operations
- Logarithms

Resources

Australian Mathematical Sciences Institute. (2011). *Fractions and the index laws in algebra*. Retrieved from http://www.amsi.org.au/teacher_modules/pdfs

Australian Mathematical Sciences Institute. (2011). *Multiples, factors and powers*. Retrieved from http://www.amsi.org.au/teacher_modules/pdfs

Muschla, J. A., Muschla, G. R., Muschla, E. (2011). *The algebra teacher's guide to reteaching essential concepts and skills.* San Francisco: Jossey-Bass