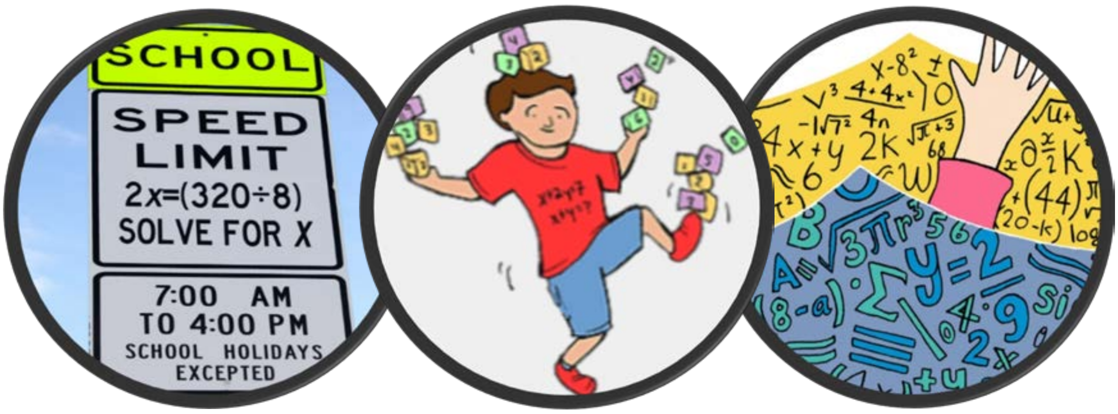
Maths Module 5

Algebra Basics

This module covers concepts such as:

- algebra rules
- collecting like terms
- simplifying equations



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Module 5

Algebra Basics

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1. What is Algebra

Algebraic thinking spans all areas of mathematics. It involves forming and recognising number relationships, and then expressing these relationships through a symbol system. For example, in words we could say that, 'An odd number is any number that when divided by two will leave one'. We could also write this as an algebraic expression:

"When $\frac{x}{2}$ has a remainder of 1, x is an odd number."

Hence, algebra provides the written form to express mathematical ideas. For instance, if I had a bag of apples that were to be shared between four people, then I could use a **pronumeral** to represent the number of apples (in this case I use the letter n), and then I can express the process of sharing n apples between four people mathematically; each person will receive $\frac{n}{4}$ apples.

AN EXAMPLE:

Algebra is best shown in a simple example:

> I just had a new part fitted to my car by the mechanic. The part cost \$270.00 dollars. The entire bill was \$342.00. How much was the cost of labour?
>
> From general knowledge you can work out that the labour must be the total cost minus the cost of the part. $\$342 - \$270$ = cost of labour.
>
> $\therefore$ \$72 = cost of labour.

But this required us to rearrange the problem in our head. (Which is easy for simple problems)

If we write the question logically, we can also show that: The part cost plus the labour cost equals cost total. So $\$270 + cost\ of\ the\ part = \$342$ (Now represent the unknown with a symbol/variable)

Now we write it symbolically: $270 + x = 342$ (we have an algebraic equation)

To solve the equation we are required to work out a value to represent $'x'$. The $'x'$ symbolises a value in the equation. In other words, 270 + something = 342. We can take guesses to calculate the answer or we can rearrange the problem and then work to balance both sides of the 'equals' sign.

$$270 + x = 342$$
$$(270 - 270) + x = 342 - 270$$
$$x = 72$$
$$\therefore The\ cost\ of labour\ was\ \$72.00$$

You will probably agree that it is easier to do the above problem in your head much faster than the method used. But for complex problems, it is important to have a step-by-step method. In other words, algebraic thinking involves working systematically to solve and abstract mathematical procedure.

1. Your Turn

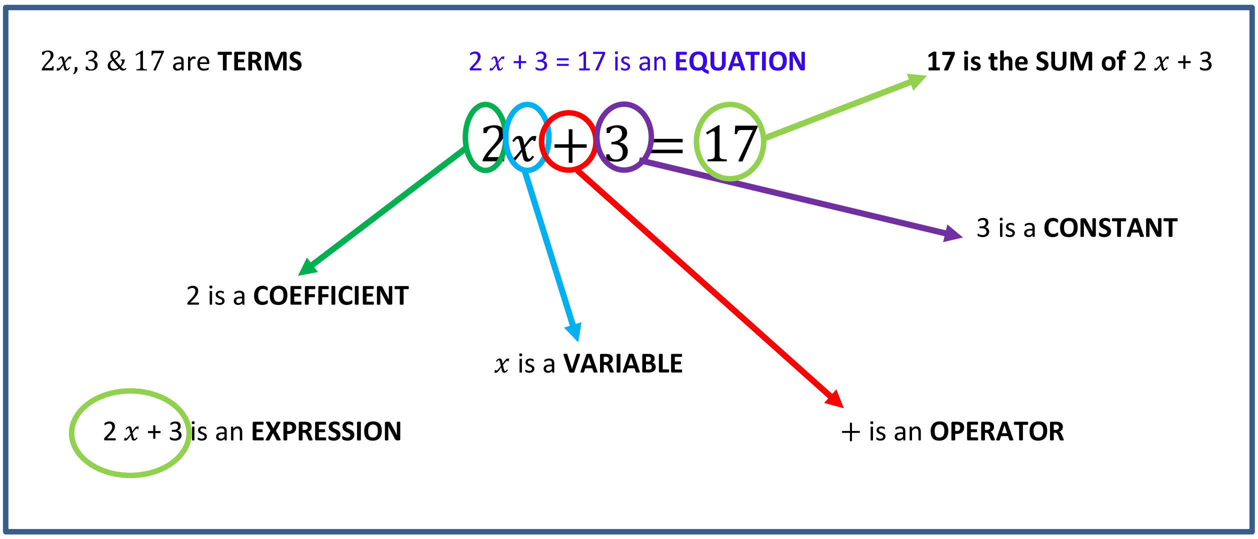
If $x = 6$, then what is the value of?

a. $4x + 3$

b. $9 - \frac{x}{2}$

c. $\frac{x}{3} + 2$

2. Glossary



Equation:	Is a mathematical sentence. It contains an equal sign meaning that both sides are equivalent.
Expression:	An algebraic expression involves numbers, operation signs, brackets/parenthesis and pronumerals that substitute numbers.
Operator:	The operation $(+, -, \times, \div)$ which separates the terms.
Term:	Parts of an expression separated by operators which could be a number, variable or product of numbers and variables. Thus $2x, 3$ & 17
Pronumeral:	A symbol that stands for a particular value.
Variable:	A letter which represents an unknown number. Most common is x, but it can be any symbol.
Constant:	Terms that contain only numbers that always have the same value.
Coefficient:	Is a number that is partnered with a variable. Between the coefficient and the variable is a multiplication. Coefficients of 1 are not shown.

In summary:

Pronumeral:	x	Operator:	$+$
Variable:	x	Terms:	$3, 2x$ (a term with 2 factors) & 17
Constant:	3		
Equation:	$2x + 3 = 17$	Left hand expression:	$2x + 3$
Coefficient:	2	Right hand expression	17 (which is the sum of the LHE)

2. Your Turn:

Complete the following for the equation: $5a + 3 = 38$

Pronumeral:		Operator:	
Variable:		Terms:	
Constant:		Expression:	
Equation:		Left hand expression:	
Coefficient:		Right hand expression	

3. Some Algebra Rules

Expressions with zeros and ones:

Zeros and ones can be eliminated. For example:

When we add zero it does not change the number, $x + 0 = x$ or $x - 0 = x$

$$(6 + 0 = 6, \quad 6 - 0 = 6)$$

If we multiply by one, then the number stays the same, $x \times 1 = x$ or $\frac{x}{1} = x$

$$(6 \times 1 = 6, \ \frac{6}{1} = 6)$$

Note: when we work with indices (powers) any number raised to the power zero is 1, this works because when we divide indices we subtract the indexes and thus get zero.

$$2^2 \div 2^2 = 4 \div 4 = 1 \quad therefore,\ 2^2 \div 2^2 = 2^{2-2} = 2^0 = 1$$

- Multiplicative Property: $1 \times x = x$
- Multiplying any number by one makes no difference.
- Additive Inverse: $x + (-x) = 0$
- Any number added to its negative equals zero.
- Multiplicative Inverse: $\frac{x \times 1}{x} = 1$
- Any number multiplied by its reciprocal equals one. $x \times \frac{1}{x} = 1;\ 4 \times \frac{1}{4} = 1$
- Symmetric Property: $If\ x = y\ then\ y = x$ Perfect harmony.
- Transitive Property: $If\ x = y\ and\ y = z, then\ x = z$

  For example, if apples cost \$2 and oranges cost \$2 then apples and oranges are the same price.

The **Calculation Priority Sequence** recognised by the mnemonic known as the BIDMAS of BODMAS.

Also a **golden rule**:
"What we do to one side we do to the other"

- Step One: $6a = 18$
- Step Two: $6a = 18$ ($\div 6$, $\div 6$)
- Step Three: $6a = 18 \div 6$
- Step Four: $a = 3$

3. Your Turn:

Here are some revision examples for practise:

a. $10 - 2 \times 5 + 1 =$

b. $10 \times 5 \div 2 - 3 =$

c. $12 \times 2 - 2 \times 7 =$

d. $48 \div 6 \times 2 - 4 =$

e. $what\ is\ the\ missing\ operation\ symbol\ 18\ \_\ 3 \times 2 + 2 = 14$

4. Addition & Multiplication Properties

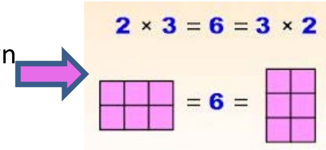
The Commutative Property: is the property where you can move things around, they commute.

The Commutative Property of Addition: $x + y = y + x$

The order that numbers are added does not affect the 'sum'; for example, $2 + 3 = 3 + 2$

The Commutative Property of Multiplication: $x \times y = y \times x$

The order that numbers are added does not affect the 'product'; as shown



The Associative Property

The Associative Property of Addition: $(x + y) + z = x + (y + z)$

When adding two or more numbers, the order you add numbers does not matter. The significance of this property is that it is possible to **regroup** the numbers so that the order of calculation can be established. For example, while $(3 + 4) + 6 = 13$ *and* $3 + (4 + 6) = 13$, the brackets signify which group of digits should be added together first.

When understanding this concept it can make working with numbers easier, because we can group 'nice' numbers together, such as the rainbow facts, number pairs that equal 10, $(4 + 6)$.

The Associative Property of Multiplication: $(x \times y) \times z = x \times (y \times z)$

When multiplying two or more numbers (factors), the order or multiplication does not matter. The significance of this property is that it is possible to **regroup** the numbers, so that the order of calculation can be established. Take a minute to compare the associative property of multiplication with the commutative property of multiplication. How do these properties differ? You might notice that in commutative property the numbers are moved around – **not** regrouped.

The Distributive Property: multiplication distributes over addition or subtraction through the brackets (parentheses). For example: $x(y + z) = xy + xz$

Allows for a separate factor to be multiplied by other factors inside the brackets individually and still get the same result. Thus every term inside the brackets is multiplied by the term that is outside of the brackets – think of this as expanding a mathematical expression.

For example:

$$2(3 + 4) = 2 \times 3 + 2 \times 4$$
$$2(7) = 6 + 8$$
$$14 = 14$$

Remember the calculation priority sequence too (BIMDAS), because if we simply worked from left to right, the answer would be 32 and thus incorrect.

Distributive Law can work for division too – ***but*** only from right to left.

For example:

$$(12 + 8) \div 2 = 10$$
$$(12 \div 2) + (8 \div 2) = 6 + 4 = 10$$

However, this will not work for $40 \div (3 + 2)$

$$40 \div (3 + 2) = 40 \div 5 = 8 \quad ✓$$
$$whereas (40 \div 3) + (40 \div 2) = 13\tfrac{1}{3} + 20 = 33\tfrac{1}{3} \quad ✗$$

4. Your Turn

a. Rewrite $3 \times 2 \times x$ by using the 'commutative property'

b. Rearrange $2(4x)$ in using the 'associative property'

c. Rewrite $8(2 + x)$ using the 'distributive property'

5. Collecting Like Terms

Algebraic thinking involves simplifying problems to make them easier to solve. Often the sight of variables can raise mathematical anxieties, yet once we understand a little more, the anxieties can dissipate.

Mathematical anxiety is common and the best way to relieve it is to recognise what we do know; then begin to work methodically to solve the problem. For instance, in the equation below, we can look at the equation as bits of information so as the equation becomes easier to solve. In short, we simplify the problem into a smaller problem and this is done by **collecting like terms**.

$$7x + 2x + 3x - 6x + 2 = 14$$

A **like term** is a term which has the same variable (which may also have the same power/exponent/index), only the coefficient is different. For example, in the equation above we can see that we have four different coefficients $(7, 2, 3, \& \ 6)$ with the same variable, x, yet there are no exponents to consider. Once identified, we can collect these like terms: $7x + 2x + 3x - 6x$

We can add and subtract the coefficients as separate from the variable: $7 + 2 + 3 - 6 = 6$
Thus, $7x + 2x + 3x - 6x = 6x$
The original equation simplifies to $6x + 2 = 14$ (adding in the constant)
Now we solve the equation:

$$6x + 2(-2) = 14 - 2$$
$$6x = 12$$
$$6\,x(\div 6) = 12 \ \div 6$$
$$x = 2$$

This mathematical process will be investigated more deeply in section 6.

EXAMPLE PROBLEM:

1. Collect the like terms and simplify:

$$5x + 3xy + 2y - 2yx + 3y^2$$

Step 1: Recognise the like terms (note: xy is the same as yx; commutative property)

$$5x + 3xy + 2y - 2yx + 3y^2$$

Step 2: Arrange the expression so that the like terms are together (remember to take the operator with the term).

$$5x + 2y + 3xy - 2yx + 3y^2$$

Step 3: Complete the operation: $5x + 2y + xy + 3y^2$
Note: a coefficient of 1 is not usually shown $\therefore$ $5x + 2y + xy + 3y^2$

5. Your Turn:

Collect the like terms using the steps above:

a. $3x + 2y - x$

b. $2x^2 - 3x^3 - x^2 + 2x$

c. $3m + 2n + 3n - m - 7$

d. $4(x + 7) + 3(2x - 2)$

e. $3(m + 2n) + 4(2m + n)$

f. $\frac{x}{3} + \frac{x}{4}$

6. Simplifying Equations: Using Expansion

To simplify equations involves 'expanding' or 'factorising'. This section helps you to investigate the concept of what it means to expand an expression. When we expand an expression, we remove the brackets, often referred to as the grouping symbols. This 'expansion' involves applying the **distributive property**.

Let's illustrate with an example:

> $x(6+9)$ As we know from the distributive law, the $'x'$ outside of the brackets is multiplied **through** the brackets. So we can express $x(6+9)$ as: $6x+9x$. In this expression we have two like terms, so we can simplify further to $15x$.

If we multiply two numbers together, then the order in which we multiply is irrelevant; **commutative property.** For example: Simplify $4(3x)$

> This could be written as $4 \times (3 \times x)$ and then as $(4 \times 3) \times x$
> Therefore, we can simplify to $12x$

6. Your Turn:

Simplify these expressions using expansion:

a. $x(4+3)$

b. $x(3-1)$

c. $x(8+6)$

Sometimes there may be **nested** grouping symbols. This happens when there are two sets of brackets – one is *nested* inside the other. This means that the operations in the inner set must be worked first.

EXAMPLE PROBLEMS:

1) $20-[3(14-10)] =$
$20-[3 \times (14-10)] =$
$20-[3 \times 4] =$
$20-12 = 8$

2) $4[(6+3) \times 5] =$
$4[9 \times 5] =$
$4(45) = 180$

3) $\frac{(3+21)}{[12-(11-7)]} = \frac{(3+21)}{(12-4)}) =$
$\frac{24}{8} = 3$

6. Your Turn

d. $8[9-(5+2)] =$

e. $2[4+5(6-5)] =$

f. $2[3(13-8) \times 4] =$

g. $\frac{2(4+6)}{2+3} =$

h. $\frac{(12 \times 4)}{4(4+2)} =$

i. What is the missing number?
$5+\{4[\_+3(7+2)]\} = 125$

7. Simplifying Equations with Exponents

Now let's work with an algebraic expression with brackets and powers. We will simplify using expansion (by removing the brackets), eventually incorporating terms with exponents:

Simplify $(3x)(6x)$

This is essentially: $(3 \times x) \times (6 \times x)$

We can change the order $(3 \times 6) \times (x \times x)$

Therefore, further simplify to $18x^2$

Work strategically:

Is $2x^2$ the same as $(2x)^2$?

Let's investigate:

$2x^2$ is $2 \times x \times x$

so if x was 5; $2 \times 5 \times 5 = 50$

$(2x)^2$ is $(2 \times x) \times (2 \times x)$

so if x was 5; $10 \times 10 = 100$

Therefore, $2x^2$ is not the same as $(2x)^2$

This example demonstrates how the **commutative property** can be applied

$$
\begin{aligned}
6x \times 2y \times 3xy &= 6 \times x \times 2 \times y \times 3 \times x \times y \\
&= 6 \times 2 \times 3 \times x \times x \times y \times y \\
&= 36x^2\, y^2
\end{aligned}
$$

…and now we also apply the 'INDEX LAWS' to simplify an expression including terms with indices (exponents) (refer to Module 4).

Simplify $5x^2 \times 6x^5$

So we can say that we have $(5 \times 6) \times (x \times x) \times (x \times x \times x \times x \times x)$

This is $30 \times x^{(2+5)}$

Therefore, $30x^7$

(Recall the first index law from Module 4)

There are steps to follow which may assist your reasoning process

Recap:

- Simplify expressions that have grouping symbols first and work from the innermost to the outer. As you do this, apply the **Calculation Priority Sequence** (BIDMAS).
    - Simplify powers
    - Multiply in order from left to right
    - Add and subtract in order form left to right.
- Then work backwards to check

EXAMPLE PROBLEMS:

- What is the difference between the expressions: $(6x)(5x)$ and $(6x) + (5x)$?
    - $(6x)(5x) = 30x^2$ whereas $(6x) + (5x) = 11x$
- One more: $(-5a^2)(-2a) = 10a^3$

  Here we apply the index law principle that $a^2 \times a^1 = a^3$ as well as the rule that a negative multiplied by a negative is positive: $(-5) \times (-2) = 10$

7. Your Turn

a. $2x + 7x + 11x =$

b. $4xy + 7xy =$

c. $6x^2 - 5x^2 =$

d. $5x^2 + 7x + 3x =$

e. $8x^2\text{y} + 2x^2y + 5xy =$

f. ...and a challenge: What is the missing number $\frac{2(\quad +4)^2}{5(14-3^2)} = 8$

8. Answers

1. a. 27 b. 6 c. 4

2.

Pronumeral:	a	Operator:	$+$
Variable:	a	Term:	$5a$ & 3
Constant:	3	Expression:	$5a + 3$
Equation:	$5a + 3 = 38$	Left hand expression:	$5a + 3$
Coefficient:	5	Right hand expression	38

3. a. 1 b. 22 c. 10 d. 12 e. –

4. a. $3 \times 2x$ *or* $2 \times 3x$; ($6x$ simplified) b. $2x \times 8$ ($16x$ simplified) c. $(8 \times 2) + (8x)$ *or* $(16 + 8x)$

5. a. $3x + 2y - x = 2x + 2y$
   b. $2x^2 - 3x^3 - x^2 + 2x = x^2 - 3x^3 + 2x$
   c. $3m + 2n + 3n - m - 7 = 2m + 5n - 7$
   d. $4(x + 7) + 3(2x - 2) = 4x + 28 + 6x - 6$
   $= 10x + 22$
   e. $3(m + 2n) + 4(2m + n) = 3m + 6n + 8m + 4n$
   $= 11m + 10n$
   f. $\frac{x}{3} + \frac{x}{4} = \frac{(4x+3x)}{12} = \frac{7x}{12}$

6. a. $4x + 3x, or\ 7x$ b. $3x - x, or\ 2x$ c. $8x + 6x, or\ 14x$ d. 16 e. 18 f. 120 g. 4 h. 2 i. 3

7. a. $20x$ b. $11xy$ c. x^2 d. $5x^2$+$10x$ e. $10x^2y + 5xy$ f. 6

9. Helpful Websites

BIDMAS: http://www.educationquizzes.com/gcse/maths/bidmas-f/

Commutative property: http://www.mathematicsdictionary.com/english/vmd/full/c/vepropertyofmultiplication.htm
http://www.purplemath.com/modules/numbprop2.htm

Like Terms: http://www.freemathhelp.com/combining-like-terms.html